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On topological *groups/vector spaces* whose closed *subgroups/vector subspaces* are separable

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Abstract

We denote by CSS the class of Hausdorff topological groups whose closed subgroups are separable; similarly CVS denotes the class of Hausdorff topological vector spaces whose closed vector subspaces are separable. It follows from the theorem of W. Comfort and G. Itzkowitz that every locally compact separable group is in CSS. From another hand, it is known that the product of continuum many copies of the reals $\mathbb{R}^{\mathfrak{c}}$ is in CVS, as a topological vector space; but $\mathbb{R}^{\mathfrak{c}}$ is not in CSS, as a topological group.

In our talk we survey the results about the classes of spaces CSS and CVS which are published recently in three author's joint papers [1-3].

Example 1. [2] *Let $\mathbb{M}$ be the Michael line. Then $C_p(\mathbb{M})$ is not in CVS while $C_p(\mathbb{M})$ is separable.*

Recall that a topological group G is called *feathered* if it contains a nonempty compact subset with a countable neighbourhood base in G.

Theorem 2. [1] *Every separable feathered group is in CSS.*

K.H. Hofmann and S.A. Morris introduced the class of *pro-Lie groups* which consists of projective limits of finite-dimensional Lie groups and proved that it contains all compact groups, all locally compact abelian groups, all connected locally compact groups and is closed under the products and closed subgroups.

Theorem 3. [1] *Let an almost connected pro-Lie group G be homeomorphic to a subspace of a separable Hausdorff space, then G is separable.*

Remark. $\mathbb{R}^{\mathfrak{c}} \notin CSS$, *so not every separable connected pro-Lie group is in CSS.*

Theorem 4. [1] *Any precompact (abelian) topological group of weight less than or equal to $\mathfrak{c}$ is topologically isomorphic to a closed subgroup of a separable pseudocompact (abelian) group of weight $\mathfrak{c}$.*

We present also an example under the CH of a separable countably compact abelian group which contains a non-separable closed subgroup.

Especially we investigated the productivity properties of the defined above classes.

Theorem 5. [3] *Let G be a separable compact group. Then the product $G \times H$ is in CSS for every $H \in CSS$.*

We don't know if Theorem 5 is valid for any separable locally compact group G. Is it true for any separable metrizable group G? for $G = \mathbb{R}$?

Theorem 6. [3] *Assume that* $2^{\omega_1} = \mathfrak{c}$. *Then there exist*:
(a) *pseudocompact topological abelian groups* G *and* H *which both are in* CSS, *but its product* $G \times H$ *is not in* CSS.
(b) *pseudocomplete locally convex vector spaces* K *and* L *which both are in* CVS, *but its product* $K \times L$ *is not in* CVS.

We don't know if such pairs of spaces as in Theoerem 6 exist in ZFC only. Recently M. Tkachenko with his coauthors showed that in ZFC there are precompact topological abelian groups G and H which both are in CSS, but its product $G \times H$ is not in CSS.

References

[1] A. Leiderman, S. Morris, M. Tkachenko. Density character of subgroups of topological groups, *Transactions of AMS*, 369 (2017), N 8, p. 5645–5664.

[2] A. Leiderman, J. Kakol, S. Morris. Nonseparable closed vector subspaces of separable topological vector spaces, *Monatshefte für Mathematik*, 182 (2017), N 1, p. 39–47.

[3] A. Leiderman, M. Tkachenko. Products of topological groups in which all closed subgroups are separable, *Topology and its Applications*, 241 (2018), p. 89–101.